

# A finite set of equilibria for the indeterminacy of linear rational expectations models

Jean-Bernard Chatelain\* and Kirsten Ralf†

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## Abstract

This paper demonstrates the existence of a finite set of equilibria in the case of the indeterminacy of linear rational expectations models. The number of equilibria corresponds to the number of ways to select  $n$  eigenvectors among a larger set of eigenvectors related to stable eigenvalues. A finite set of equilibria is a substitute to continuous (uncountable) sets of sunspots equilibria, when the number of independent eigenvectors for each stable eigenvalue is equal to one.

**JEL classification numbers:** C60, C61, C62, E13, E60.

**Keywords:** Linear rational expectations models, indeterminacy, multiple equilibria, Riccati equation, sunspots.

*"Das kann als Riccatische gleichung des matrizenkalküls angesehen werden."* Radon (1928) p.190.

## 1 Introduction

This paper demonstrates that there is a finite set of rational expectations equilibria in the case of indeterminacy for linear rational expectations models, which is a *substitute* to uncountable (continuously infinite) sets of sunspots

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\*Paris School of Economics, Université Paris I Pantheon Sorbonne, CES, Centre d'Economie de la Sorbonne, 106-112 Boulevard de l'Hôpital 75647 Paris Cedex 13. Email: jean-bernard.chatelain@univ-paris1.fr

†ESCE International Business School, 10 rue Sextius Michel, 75015 Paris, Email: Kirsten.Ralf@esce.fr.

equilibria (Gourieroux et al. (1982)). This occurs when the number of independent eigenvectors for each stable eigenvalue is equal to one, in particular, when all stable eigenvalues are distinct. This paper extends Blake and Kirsanova (2012) results for time-consistent optimal policy rules to the general case of Blanchard Kahn (1980) solutions.

Blanchard and Kahn (1980) states that there are multiple equilibria with rational expectations (or indeterminacy) when the number  $n$  of pre-determined variables is lower than the number  $s$  of eigenvalues below one in absolute values. In this case, the initial values of the number  $m$  of non pre-determined "forward" variables may be driven by continuous random variables of zero mean, independently and identically distributed over time (Gourieroux *et al.* (1982)).

Besides this continuous infinity of sunspots equilibria, it is feasible to extend the computation of saddlepath unique rational expectations equilibrium (Blanchard and Kahn (1980), Boucekkine and Le Van (1996)) to the case of multiple equilibria. These rational expectations equilibria are solutions of a matrix Riccati equation (Radon (1928), Le Van (1986), Abou-Kandil *et al.* (2003)). This paper demonstrates that there is a finite a number of equilibria, at most equal to  $\frac{s!}{n!(s-n)!}$ . This is the number of ways to choose  $n$  distinct eigenvectors among a larger set of  $s$  eigenvectors related to eigenvalues with absolute values below one, when there is only one independent eigenvector for each of these eigenvalues.

## 2 A finite set of equilibria with indeterminacy

Blanchard and Kahn (1980) consider a linear rational expectations model:

$$\begin{pmatrix} \mathbf{k}_{t+1} \\ {}_t\mathbf{q}_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{A}_{nn} & \mathbf{A}_{nm} \\ \mathbf{A}_{mn} & \mathbf{A}_{mm} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} + \gamma \mathbf{z}_t \quad (1)$$

where  $\mathbf{k}_t$  is an  $(n \times 1)$  vector of variables predetermined at  $t$  with initial conditions  $\mathbf{k}_0$  given (shocks can straightforwardly be included into this vector);  $\mathbf{q}$  is an  $(m \times 1)$  vector of variables non-predetermined at  $t$ ;  $\mathbf{z}$  is an  $(k \times 1)$  vector of exogenous variables;  $\mathbf{A}$  is  $(n + m) \times (n + m)$  matrix,  $\gamma$  is a  $(n + m) \times k$  matrix,  ${}_t\mathbf{q}_t$  is the agents expectations of  $\mathbf{q}_{t+1}$  defined as follows:

$${}_t\mathbf{q}_{t+1} = E_t(\mathbf{q}_{t+1} | \Omega_t). \quad (2)$$

$\Omega_t$  is the information set at date  $t$  (it includes past and current values of all endogenous variables and may include future values of exogenous variables). A **predetermined** variable is a function only of variables known at date  $t$  so that  $\mathbf{k}_{t+1} = {}_t\mathbf{k}_{t+1}$  whatever the realization of the variables in  $\Omega_{t+1}$ . A **non-predetermined** variable can be a function of any variable in  $\Omega_{t+1}$ , so that we can conclude that  $\mathbf{q}_{t+1} = {}_t\mathbf{q}_{t+1}$  only if the realization of all variables in  $\Omega_{t+1}$  are equal to their expectations conditional on  $\Omega_t$ .

Boundary conditions for the policy-maker's first order conditions are the given initial conditions for predetermined variables  $\mathbf{k}_0$  and Blanchard and Kahn (1980) hypothesis ruling out the exponential growth of the expectations of  $\mathbf{w} = (\mathbf{k}, \mathbf{q}, \mathbf{z})$ :

$$\forall t \in \mathbb{N}, \exists \bar{\mathbf{w}}_t \in \mathbb{R}^k, \exists \theta_t \in \mathbb{R}, \text{ such that } |E_t(\mathbf{w}_{t+1} | \Omega_t)| \leq (1 + i)^{\theta_t} \bar{\mathbf{w}}_t, \forall i \in \mathbb{R}^+. \quad (3)$$

**Definition:** Besides other sunspots equilibria (Gourieroux et al. [1982]), let us define a set of rational expectations solutions, which are such that non predetermined variables are a linear function of pre-determined variables, where the matrix  $\mathbf{N}_{mn}$  is to be found, and with bounded solutions for pre-determined variables, so that the eigenvalues  $\lambda_i$  of the matrix  $\mathbf{A}_{nn} - \mathbf{A}_{nm}\mathbf{N}_{mn}$  are below one ("stable eigenvalues"):

$$\mathbf{q}_{t+1} = -\mathbf{N}_{mn}\mathbf{k}_{t+1} \quad (4)$$

$$\mathbf{k}_{t+1} = (\mathbf{A}_{nn} - \mathbf{A}_{nm}\mathbf{N}_{mn}) \mathbf{k}_t \quad (5)$$

$$\lambda(\mathbf{A}_{nn} - \mathbf{A}_{nm}\mathbf{N}_{mn}) = \{\lambda_i \text{ with } |\lambda_i| < 1, i \in \{1, \dots, n\}\} \quad (6)$$

**Proposition:**  $\mathbf{A}$  has  $s$  stable eigenvalues and  $n + m - s$  unstable eigenvalues.

*Case 1. When  $0 \leq s < n$ , the number of stable eigenvalues is strictly below the number of pre-determined variables, there is no rational expectations equilibrium (Blanchard and Kahn (1980)).*

*Case 2. When  $s = n$ , the number of stable eigenvalues is strictly equal to the number of predetermined variables, there is a unique rational expectations equilibrium (Blanchard and Kahn (1980)).*

*Case 3. When  $n < s \leq n + m$ , the number of rational expectations equilibria defined above is given by the number of ways of selecting  $n$  independent (right column) eigenvectors  $\begin{pmatrix} \mathbf{P}_{nn} \\ \mathbf{P}_{mn} \end{pmatrix}$  among a larger set of independent eigenvectors related to stable eigenvalues. If  $\mathbf{P}_{nn}$  is invertible, they*

corresponds to the number of rational expectations equilibria determined by each matrix  $\mathbf{N}_{mn} = -\mathbf{P}_{mn}\mathbf{P}_{nn}^{-1}$ :

**Case 3.1. Finite number of equilibria.** If the number of independent eigenvectors (geometric multiplicity) of each stable eigenvalues of  $\mathbf{A}$  is exactly one, the number of equilibria is given by  $\frac{s_1!}{n!s_1!}$  where the number of stable eigenvalues not counting their multiplicity is denoted  $s_1 \leq s$ . In particular, if all the stable eigenvalues of  $\mathbf{A}$  are distinct, then the number of equilibria is  $\frac{s!}{n!s!}$ .

**Case 3.2. Uncountable number of equilibria.** If there is at least one stable eigenvalue of  $\mathbf{A}$  with its number of independent eigenvectors (geometric multiplicity) which is at least equal to two, then, there always exists an uncountable number of equilibria. This condition for an uncountable number of equilibria is distinct from e.g. Gourieroux et al. (1982).

For example, for  $n = 1$ ,  $m = 1$ , and with a unique stable eigenvalue  $\lambda_1$  with two independent column vectors  $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$ , there is an uncountable number of single eigenvectors  $\mathbf{P}_\alpha = \mathbf{P}_1 + \alpha\mathbf{P}_2$  with  $\alpha \in \mathbb{C}$  leading to solutions  $\mathbf{N}_{mn,\alpha} = -\mathbf{P}_{mn,\alpha}\mathbf{P}_{nn,\alpha}^{-1}$ . For  $n = 2$ ,  $m = 1$ , including another eigenvalue  $\lambda_3$  with a multiplicity equal to one and an eigenvector denoted  $\mathbf{P}_3$ , there is a single case of  $n = 2$  columns eigenvector  $(\mathbf{P}_1, \mathbf{P}_2)$  and an uncountable number of  $n = 2$  eigenvector matrix  $\mathbf{P}_{3\alpha} = (\mathbf{P}_3, \mathbf{P}_\alpha)$  with  $\alpha \in \mathbb{C}$  allowing to compute solutions  $\mathbf{N}_{mn} = -\mathbf{P}_{mn}\mathbf{P}_{nn}^{-1}$  (see a numerical example for  $n = 2$ ,  $m = 2$  in Abou-Kandil et al. (2003) p.25).

**Proof:**

Let us consider a matrix  $\mathbf{N}_{mn}$  such that:

$$\begin{pmatrix} \mathbf{k}_{\mathbf{N},t} \\ \mathbf{q}_{\mathbf{N},t} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{nm} \\ -\mathbf{N}_{mn} & \mathbf{I}_m \end{pmatrix} \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix}$$

with  $\mathbf{T} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{nm} \\ -\mathbf{N}_{mn} & \mathbf{I}_m \end{pmatrix}$  and  $\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{nm} \\ \mathbf{N}_{mn} & \mathbf{I}_m \end{pmatrix}$  (7)

So that:

$$\begin{pmatrix} \mathbf{k}_{\mathbf{N},t+1} \\ \mathbf{q}_{\mathbf{N},t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{nm} \\ \mathbf{N}_{mn} & \mathbf{I}_m \end{pmatrix} \begin{pmatrix} \mathbf{A}_{nn} & \mathbf{A}_{nm} \\ \mathbf{A}_{mn} & \mathbf{A}_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{nm} \\ -\mathbf{N}_{mn} & \mathbf{I}_m \end{pmatrix} \begin{pmatrix} \mathbf{k}_{\mathbf{N},t} \\ \mathbf{q}_{\mathbf{N},t} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{k}_{\mathbf{N},t+1} \\ \mathbf{q}_{\mathbf{N},t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{nn} - \mathbf{A}_{nm}\mathbf{N}_{mn} & \mathbf{A}_{nm} \\ g(\mathbf{N}_{mn}) & \mathbf{A}_{mm} + \mathbf{N}_{mn}\mathbf{A}_{nm} \end{pmatrix} \begin{pmatrix} \mathbf{k}_{\mathbf{N},t} \\ \mathbf{q}_{\mathbf{N},t} \end{pmatrix} \text{ with (8)}$$

$$g(\mathbf{N}_{mn}) = \mathbf{A}_{mn} + \mathbf{A}_{mm}\mathbf{N}_{mn} - \mathbf{N}_{mn}\mathbf{A}_{mm} - \mathbf{N}_{mn}\mathbf{A}_{nm}\mathbf{N}_{mn} = \mathbf{0}_{mn} \quad (9)$$

$g(\mathbf{N}_{mn}) = \mathbf{0}_{mn} = \partial\mathbf{N}_{mn}/\partial t$  is a matrix equation including a constant, two linear terms and a quadratic term  $\mathbf{N}_{mn}\mathbf{A}_{nm}\mathbf{N}_{mn}$ , which Radon (1928)

denoted as matrix Riccati extension of scalar Riccati differential equations. If  $\mathbf{N}_{mn}$  is a solution with constant coefficients of  $g(\mathbf{N}_{mn}) = \mathbf{0}_{mn}$ , then the characteristic polynomial of matrix  $\mathbf{A}$  is the product of two characteristic polynomials, as  $\det(\mathbf{T}) = 1 = \det(\mathbf{T}^{-1})$ :

$$\det(\mathbf{A} - \lambda \mathbf{I}_{n+m}) = \det(\mathbf{A}_{nn} - \mathbf{A}_{nm} \mathbf{N}_{mn} - \lambda \mathbf{I}_n) \cdot \det(\mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm} - \lambda \mathbf{I}_m) = 0 \quad (10)$$

Each solution  $\mathbf{N}_{mn}$  of  $g(\mathbf{N}_{mn}) = \mathbf{0}_{mn}$  corresponds to a particular partition of the eigenvalues of the matrix  $\mathbf{A}$  since its eigenvalues are exactly the eigenvalues of  $\mathbf{A}_{nn} - \mathbf{A}_{nm} \mathbf{N}_{mn}$  (with  $n$  eigenvalues counting multiplicity) and  $\mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm}$  (with  $m$  eigenvalues counting multiplicity). A Jordan canonical transformation  $\mathbf{J}$  of the  $\mathbf{A}$  matrix with  $\mathbf{P}$  a matrix of right eigenvectors is:

$$\begin{pmatrix} \mathbf{A}_{nn} & \mathbf{A}_{nm} \\ \mathbf{A}_{mn} & \mathbf{A}_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{P}_{nn} & \mathbf{P}_{nm} \\ \mathbf{P}_{mn} & \mathbf{P}_{mm} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{nn} & \mathbf{P}_{nm} \\ \mathbf{P}_{mn} & \mathbf{P}_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{J}_n & \mathbf{0}_{nm} \\ \mathbf{0}_{mn} & \mathbf{J}_m \end{pmatrix} \quad (11)$$

where  $\mathbf{J}_{nn}$  is a  $n \times n$  Jordan matrix with the eigenvalues of  $\mathbf{A}_{nn} - \mathbf{A}_{nm} \mathbf{N}_{mn}$  and  $\mathbf{J}_{mm}$  is a  $m \times m$  Jordan matrix with the eigenvalues of  $\mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm}$ . One has:

$$\begin{pmatrix} \mathbf{A}_{nn} - \mathbf{A}_{nm} \mathbf{N}_{mn} & \mathbf{A}_{nm} \\ \mathbf{0}_{mn} & \mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm} \end{pmatrix} \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{nm} \\ -\mathbf{N}_{mn} & \mathbf{I}_m \end{pmatrix} \begin{pmatrix} \mathbf{P}_{nn} & \mathbf{P}_{nm} \\ \mathbf{P}_{mn} & \mathbf{P}_{mm} \end{pmatrix} \\ = \begin{pmatrix} \mathbf{I}_n & \mathbf{0}_{nm} \\ -\mathbf{N}_{mn} & \mathbf{I}_m \end{pmatrix} \begin{pmatrix} \mathbf{P}_{nn} & \mathbf{P}_{nm} \\ \mathbf{P}_{mn} & \mathbf{P}_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{J}_n & \mathbf{0}_{nm} \\ \mathbf{0}_{mn} & \mathbf{J}_m \end{pmatrix} \quad (12)$$

which implies:

$$\begin{pmatrix} (\mathbf{A}_{nn} - \mathbf{A}_{nm} \mathbf{N}_{mn}) \mathbf{P}_{nn} + \mathbf{A}_{nm} (\mathbf{P}_{mn} - \mathbf{N}_{mn} \mathbf{P}_{nn}) & * \\ (\mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm}) (\mathbf{P}_{mn} - \mathbf{N}_{mn} \mathbf{P}_{nn}) & * \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{nn} \mathbf{J}_{nn} & * \\ (\mathbf{P}_{mn} - \mathbf{N}_{mn} \mathbf{P}_{nn}) \mathbf{J}_{nn} & * \end{pmatrix} \quad (13)$$

Because the eigenvalues of  $\mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm}$  are not the eigenvalues of  $\mathbf{J}_{nn}$ , then  $(\mathbf{P}_{mn} - \mathbf{N}_{mn} \mathbf{P}_{nn})$  cannot stack eigenvectors (each of them distinct from the zero vector by definition) of  $\mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm}$ . Then, the second equality for block matrices ( $i = 2, j = 1$ ) is valid:

$$(\mathbf{A}_{mm} + \mathbf{N}_{mn} \mathbf{A}_{nm}) (\mathbf{P}_{mn} - \mathbf{N}_{mn} \mathbf{P}_{nn}) = (\mathbf{P}_{mn} - \mathbf{N}_{mn} \mathbf{P}_{nn}) \mathbf{J}_{nn} \quad (14)$$

only and only if  $\mathbf{P}_{mn} - \mathbf{N}_{mn} \mathbf{P}_{nn} = \mathbf{0}$ . Then, if  $\mathbf{P}_{nn}$  is invertible, one finds the solutions  $\mathbf{N}_{mn} = -\mathbf{P}_{mn} \mathbf{P}_{nn}^{-1}$ .

According to the rational expectations equilibria definition, one needs to find at least  $n$  stable eigenvalues, and compute  $\mathbf{N}_{mn}$  using a set of  $n$  column eigenvectors  $\begin{pmatrix} \mathbf{P}_{nn} \\ \mathbf{P}_{mn} \end{pmatrix}$  related to these stable eigenvalues. The number of rational expectations equilibria is then given by the number of ways of selecting  $n$  independent (right column) eigenvectors  $\begin{pmatrix} \mathbf{P}_{nn} \\ \mathbf{P}_{mn} \end{pmatrix}$  related to the stable eigenvalues  $s \geq n$ .

Finally, the first equality for block matrices ( $i = 1, j = 1$ ) becomes:

$$(\mathbf{A}_{nn} - \mathbf{A}_{nm}\mathbf{N}_{mn})\mathbf{P}_{nn} = \mathbf{P}_{nn}\mathbf{J}_{nn} \quad (15)$$

Hence, the matrix  $\mathbf{P}_{nn}$  is an eigenvectors matrix of the matrix  $\mathbf{A}_{nn} - \mathbf{A}_{nm}\mathbf{N}_{mn}$ . **Q.E.D.**

A similar demonstration with transpose matrices holds for left row eigenvectors  $\begin{pmatrix} \mathbf{Q}_{mn} \\ \mathbf{Q}_{mm} \end{pmatrix}$  with  $\mathbf{Q} = \mathbf{P}^{-1}$  chosen among a set of  $s > n$  row eigenvectors related to stable eigenvalues. If  $\mathbf{Q}_{mm}$  is invertible, one finds the solutions  $\mathbf{N}_{mn} = -\mathbf{P}_{mn}\mathbf{P}_{nn}^{-1} = \mathbf{Q}_{mm}^{-1}\mathbf{Q}_{mn}$ .

### 3 Conclusion

A finite set of rational expectations equilibria (when the number of independent eigenvectors for each stable eigenvalue is equal to one) exists at each period. For a chosen equilibrium with a given set of eigenvectors at a given period to be found again on the following periods, one needs to assume that the economic agents select the same set of eigenvectors at each period. In this case, economic agents shape their rational expectations following the same procedure at each period in a time-consistent manner.

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